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Gravitational wave instabilities in the presence of photon beams

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Abstract
The kinetic theory of photons interacting with gravitational waves is developed. Non-thermal photon distributions are considered. The possibility of photon Landau damping and photon beam instabilities of gravitational waves, both in the kinetic and fluid regimes, is discussed.

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1. Introduction

The interaction of photons with the gravitational field is one of the major problems in the theory of gravitation. The effects known as the gravitational redshift and the gravitational lens are of considerable importance not only historically, but also in present day theoretical research and astronomical observations.

In a recent study [1], it was shown that such gravitational effects can be described as particular examples of photon acceleration, a concept successfully developed in plasma physics and in optics [2]. Furthermore, photon blueshift was also predicted as a result of photon interaction with a gravitational wavepacket, propagating in a plasma or in the absence of matter. These results indicate the existence of a transfer of energy between the photon field (more precisely the electromagnetic field) and the gravitational field.

In the present paper, a statistical theory of the photon interaction with a gravitational wave is developed. Starting from the photon kinetic equation in a gravitational field, the dispersion relation of a gravitational wave is derived, showing that photon Landau damping can be possible. This result extends the concept of photon Landau damping of plasma waves [3] to the case of propagation of metric waves in vacuum.

The possible occurrence of Landau damping of gravitational waves in the presence of a photon gas is not new in the literature [4–6]; but here we give a different perspective. Furthermore, the possibility of exciting gravitational waves by a photon beam is examined in two different regimes: the kinetic regime associated with photon beams with a large spectrum, corresponds to a negative Landau damping (or a positive Landau growth) and the fluid regime, associated with nearly mono-energetic photon beams, is also examined in detail. The resulting growth rates are much more favourable than those of the kinetic regime and give some hope of obtaining observable gravitational wave instabilities in astrophysical environments, or even in future ultra-intense laser installations.

2. Photon kinetic equation

Let us consider the photon kinetic equation in a curved spacetime. For particles with momentum p_i and coordinates x^i, with $i = 0, 1, 2, 3$, we can define a number density in the eight-dimensional phase space, $N\left(x^i, p_i\right)$.

In the absence of collisions between these particles, the kinetic equation can be stated as the conservation of the number density along the geodesics. Using the affine parameter λ to characterize the geodesic lines, we can write this conservation law as [7]

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} N\left(x^i(\lambda), p_i(\lambda)\right) = 0. \tag{1}$$

If the particle motion is Hamiltonian, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} = \frac{\partial}{\partial\lambda} + [H, \quad] \tag{2}$$

where H is the Hamiltonian function describing the single particle trajectories. Noting that $N\left(x^i, p_i\right)$ is not explicitly dependent on λ, we get

$$\left[H, N\left(x^i, p_i\right)\right] = \left(\frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}\right) N\left(x^i, p_i\right) = 0. \tag{3}$$

Now, if the particles are photons, we can identify H with the photon frequency, or more precisely, with the time-like component of the four-wavevector k_i, such that $p_i = \hbar k_i$. In the following, we assume $\hbar = 1$. Then we write

$$H = -k_0\left(x^0, x^\alpha, k_\alpha\right) \tag{4}$$

with $\alpha = 1, 2, 3$. Then we get

$$\left(\frac{\partial}{\partial x^0} + \frac{\partial k_0}{\partial k_\alpha}\frac{\partial}{\partial x^\alpha} - \frac{\partial k_0}{\partial x^i}\frac{\partial}{\partial k_i}\right) N\left(x^i, k_i\right) = 0. \tag{5}$$

It is also known that the photon dispersion relation is

$$k_i k^i = g^{ij} k_i k_j = 0 \tag{6}$$

where g^{ij} are the components of the metric tensor. In general, these quantities depend on x^i, which means that we can establish a relation of the form:

$$\omega = \omega\left(x^\alpha, k_\alpha, t\right). \tag{7}$$

Here we have used the frequency $\omega = -k_0 c$, and the time variable $t = x^0/c$. This expression represents the dispersion relation of the photons in vacuum and provides the Hamiltonian function (4). The explicit form of this expression is discussed below.

From the existence of such a dispersion relation we can also conclude that

$$N\left(x^i, k_i\right) = N\left(\omega, t; x^\alpha, k_\alpha\right) = 2\pi N\left(x^\alpha, k_\alpha, t\right)\delta\left(\omega - \omega\left(x^\alpha, k_\alpha, t\right)\right). \tag{8}$$

After integration over ω, we obtain from equation (5)

$$\left(\frac{\partial}{\partial t} + \frac{\partial\omega}{\partial k_\alpha}\frac{\partial}{\partial x^\alpha} - \frac{\partial\omega}{\partial x^\alpha}\frac{\partial}{\partial k_\alpha}\right) N\left(x^\alpha, k_\alpha, t\right) = 0. \tag{9}$$

This is the final expression for the photon kinetic equation in a curved spacetime. The limiting case of a flat spacetime and its relation with the Wigner–Moyal equation for the electromagnetic field is discussed elsewhere [8]. The characteristic curves of this partial differential equation are

$$\frac{\mathrm{d}x^\alpha}{\partial t} = \frac{\partial \omega}{\partial k_\alpha} \qquad \frac{\mathrm{d}k_\alpha}{\partial t} = -\frac{\partial \omega}{\partial x^\alpha}. \tag{10}$$

These are nothing but the single photon equations of motion, written in canonical form.

3. Photon dispersion relation

Let us write equation (6) in a more explicit form. Using $\omega = -k_0 c$, and noting the existence of a symmetry in the components of the metric tensor ($g^{0\alpha} = g^{\alpha 0}$), we have

$$g^{00}\omega^2 - 2g^{0\alpha}\omega k_\alpha c + g^{\alpha\beta}k_\alpha k_\beta c^2 = 0. \tag{11}$$

Solving for the frequency ω, we obtain

$$\frac{\omega}{c} = y^{0\alpha}k_\alpha + \sqrt{\left(y^{0\alpha}k_\alpha\right)^2 - y^{\alpha\beta}k_\alpha k_\beta} \tag{12}$$

with $y^{0\alpha} = g^{0\alpha}/g^{00}$ and $y^{\alpha\beta} = g^{\alpha\beta}/g^{00}$.

Let us first assume a flat spacetime. Using the Minkowski metric tensor, we have $g^{ij} = \eta^{ij} = 1$, for $i = j = 0$, and $= -\delta^{ij}$ for $i, j = 1, 2, 3$, where δ^{ij} is the Kroenecker delta symbol. The dispersion relation (12) is then reduced to $\omega = kc$, where $k = \sqrt{\delta^{\alpha\beta}k_\alpha k_\beta}$.

If the flat spacetime is perturbed by a gravitational wave, we can use

$$g^{ij} = \eta^{ij} + h^{ij} \tag{13}$$

where $|h^{ij}| \ll 1$. In this case, we have

$$y^{0\alpha} \simeq h^{0\alpha} \qquad y^{\alpha\beta} \simeq -\delta^{\alpha\beta} + (h^{00}\delta^{\alpha\beta} + h^{\alpha\beta}). \tag{14}$$

Replacing in equation (12), we obtain

$$\frac{\omega}{c} = h^{0\alpha}k_\alpha + k\sqrt{1 + \left(h^{0\alpha}k_\alpha/k\right)^2 - (h^{00}\delta^{\alpha\beta} + h^{\alpha\beta})\left(k_\alpha k_\beta/k^2\right)}. \tag{15}$$

To the lowest order in the metric perturbation h^{ij}, this can be written as

$$\omega \simeq kc[1 + f(h^{ij})] \tag{16}$$

with

$$f(h^{ij}) = -\frac{1}{2}h^{00} + \frac{k_\alpha}{k}h^{0\alpha} - \frac{k_\alpha k_\beta}{2k^2}h^{\alpha\beta}. \tag{17}$$

This is the dispersion relation for a photon with frequency ω, moving in a perturbed empty spacetime. Alternatively, we could also write this dispersion relation as

$$\omega = \omega_0 - \Omega_{ij}h^{ij} \tag{18}$$

where $\omega_0 = kc$ and

$$\Omega_{ij} = \frac{1}{2}\delta_{i0}\delta_{i0} - \delta_{i0}\delta_j^\alpha\frac{k_\alpha}{k} + \delta_i^\alpha\delta_j^\beta\frac{k_\alpha k_\beta}{2k^2}. \tag{19}$$

Similar expressions for the photon frequency as seen by a moving massive particle in the presence of a gravitational wave were derived by Grishchuk [9].

4. Linearized kinetic equation

Let us assume that the metric perturbation h^{ij} is associated with a perturbation in the photon number density $\tilde{N}$, such that

$$N\left(x^{\alpha}, k_{\alpha}, t\right) = N_0\left(x^{\alpha}, k_{\alpha}, t\right) + \tilde{N}\left(x^{\alpha}, k_{\alpha}, t\right). \tag{20}$$

The spacetime evolution of such quantity is described by the kinetic equation (9). Linearizing this equation with respect to the perturbation, we obtain

$$\mathcal{L}_0 \tilde{N} + \mathcal{L}' N_0 = 0 \tag{21}$$

with the differential operators $\mathcal{L}_0$ and $\mathcal{L}'$ defined by:

$$\mathcal{L}_0 = \frac{\partial}{\partial t} + \frac{\partial \omega_0}{\partial k_\alpha}\frac{\partial}{\partial x^\alpha} - \frac{\partial \omega_0}{\partial x^\alpha}\frac{\partial}{\partial k_\alpha} \tag{22}$$

and

$$\mathcal{L}' = -h^{ij}\frac{\partial \Omega^{ij}}{\partial k_\alpha}\frac{\partial}{\partial x^\alpha} + \Omega^{ij}\frac{\partial h^{ij}}{\partial x^\alpha}\frac{\partial}{\partial k_\alpha}. \tag{23}$$

Here, it should be noted that according to equations (10), the components of the (unperturbed) photon velocity are determined by

$$v^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}t} = \frac{\partial \omega_0}{\partial k_\alpha} \tag{24}$$

and that in a flat spacetime, we also have

$$\frac{\partial \omega_0}{\partial x^\alpha} = c\frac{\partial k}{\partial x^\alpha} = 0. \tag{25}$$

Let us also assume that the metric tensor and photon number density perturbations, h^{ij} and $\tilde{N}$, take the elementary form $\exp \mathrm{i}\left(q_\alpha x^\alpha - \Omega t\right)$. The operator $\mathcal{L}_0$, defined above, is then reduced to

$$\mathcal{L}_0 = -\mathrm{i}\Omega + \mathrm{i}q_\alpha v^\alpha \tag{26}$$

and equation (21) becomes

$$\tilde{N} = \frac{\mathrm{i}}{\left(\Omega - q_\alpha v^\alpha\right)}\mathcal{L}' N_0. \tag{27}$$

5. Gravitational wave equation

It is well known that in the weak field approximation the Einstein field equations [10] can be reduced to

$$\Box^2 h_{ij} = -\kappa S_{ij} \tag{28}$$

where

$$\Box^2 = \eta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \delta^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}. \tag{29}$$

We have introduced $\kappa = 16\pi G/c^2$, where G is the gravitational constant, and we also have

$$S_{ij} = T_{ij} - \tfrac{1}{2}\eta_{ij}T_\lambda^\lambda. \tag{30}$$

For a gas of N particles, the energy–momentum tensor [10] would be given by

$$T^{ij}(x, t) = \sum_n^N \frac{p_n^i p_n^j}{E_n}\delta_{(3)}(x - v_n t) \tag{31}$$

where p_n, E_n and v_n are the momentum, energy and velocity of the particles, respectively.

In the case of a gas of photon, described by the photon number density $N(x^\alpha, k_\alpha, t)$, we use $p = \hbar k$, $E = \hbar\omega$, and replace the sum by an integral over the photon population

$$T^{ij}(x^\alpha, t) = \hbar \int \frac{k^i k^j}{\omega(x^\alpha, k_\alpha, t)} N(x^\alpha, k_\alpha, t)\, \mathrm{d}k^{(3)} \tag{32}$$

where $\mathrm{d}k^{(3)} = \mathrm{d}k_1 \mathrm{d}k_2 \mathrm{d}k_3/(2\pi)^3$.

From equation (20), we have a steady-state $\bar{T}^{ij}$ associated with the equilibrium distribution N_0 and a perturbed value of the energy–momentum tensor $\tilde{T}^{ij}$ associated with the perturbation $\tilde{N}$. However, the steady-state does not contribute to the radiation field, which means that the equation is reduced to

$$\Box^2 h_{ij} = -\kappa \tilde{S}_{ij}. \tag{33}$$

For a perturbation of the form $\exp \mathrm{i}\,(q_\alpha x^\alpha - \Omega t)$, we get

$$(\Omega^2 - q^2c^2)h_{ij} = \kappa c^2 \tilde{S}_{ij}. \tag{34}$$

Here we note that T^λ_λ is proportional to $k_\lambda k^\lambda$ and, according to equation (6), we conclude that $T^\lambda_\lambda = 0$. Using equations (30) and (32), we have, after linearization

$$\tilde{T}_{ij} = \hbar \int \frac{k_i k_j}{kc} \tilde{N}(x^\alpha, k_\alpha, t)\, \mathrm{d}k^{(3)}. \tag{35}$$

Equation (34) then becomes

$$(\Omega^2 - q^2c^2)h_{ij} = a \int \frac{k_i k_j}{k} \tilde{N}\, \mathrm{d}k^{(3)} \tag{36}$$

with the constant $a = \kappa\hbar c$, and using equation (20)

$$(\Omega^2 - q^2c^2)h_{ij} = \mathrm{i}a \int \frac{k_i k_j}{k} \frac{\mathcal{L}' N_0}{(\Omega - q_\alpha v^\alpha)}\, \mathrm{d}k^{(3)}. \tag{37}$$

For a homogeneous steady-state photon distribution n_0, we see from equation (23) that we simply have

$$\mathcal{L}' N_0 = \mathrm{i} q_\alpha \Omega_{\mu\nu} h^{\mu\nu} \frac{\partial N_0}{\partial k_\alpha}. \tag{38}$$

At this point, it is also useful to write

$$h^{ij} = \mathrm{e}^{ij} h \tag{39}$$

where h is the gravitational wave amplitude and e^{ij} is the unit polarization tensor, such that

$$\mathrm{e}^{ij}\mathrm{e}^*_{ij} = 1. \tag{40}$$

Replacing equations (39)–(40) in equation (37), we finally obtain

$$\Omega^2 - q^2c^2 = -a \int \frac{\left(\mathrm{e}^{ij*} k_i k_j\right)}{k} \left(\mathrm{e}^{\mu\nu}\Omega_{\mu\nu}\right) \frac{q_\alpha \partial N_0/\partial k_\alpha}{(\Omega - q_\alpha v^\alpha)}\, \mathrm{d}k^{(3)}. \tag{41}$$

This is the dispersion relation for gravitational waves with a frequency Ω, propagating in a photon gas.

6. Photon Landau damping

Let us rewrite the above dispersion relation as

$$\Omega^2 - q^2c^2 = -aI \tag{42}$$

with the integral I defined as

$$I = \int g(k_\alpha)\frac{q_\alpha \partial N_0/\partial k_\alpha}{(\Omega - q_\alpha v^\alpha)}\,\mathrm{d}k^{(3)} \tag{43}$$

and

$$g(k_\alpha) = \frac{\left(\mathrm{e}^{ij*}k_i k_j\right)}{k}\left(\mathrm{e}^{\mu\nu}\Omega_{\mu\nu}\right). \tag{44}$$

Without loss of generality, we can assume that the gravitational wave propagates along the direction x^1. This means that

$$q_\alpha = (q, 0, 0) \tag{45}$$

and

$$q_\alpha v^\alpha = (c/k)qk^1. \tag{46}$$

We also note that $k_1 = k^1$. Defining $k_\perp = (k_2, k_3)$, we can then write

$$I = \int g(k_1, k_\perp)\frac{q\partial N_0/\partial k_1}{(\Omega - qc/k)k_1}\frac{\mathrm{d}k_1\mathrm{d}k_\perp}{(2\pi)^3} \tag{47}$$

or, equivalently,

$$I = -\int \frac{\mathrm{d}k_\perp}{(2\pi)^2}\int \frac{\mathrm{d}k_1}{2\pi} g(k_1, k_\perp)\frac{q\partial N_0/\partial k_1}{(u - \Omega/q)} \tag{48}$$

with

$$u = \frac{c}{k}k_1 = ck_1\left(k_1^2 + k_\perp^2\right)^{-1/2}. \tag{49}$$

This expression for the integral shows that there is a resonant value for the photon velocity u_R, such that

$$u_R(k_1, k_\perp) = \Omega/q. \tag{50}$$

Developing equation (49) around this resonant value, we can write

$$u(k_1, k_\perp) \simeq u_r + (k_1 - k_R)\left(\frac{\partial u}{\partial k_1}\right)_{k_1 = k_R} \tag{51}$$

where k_R is the value of k_1 that satisfies, for a generic $k_\perp$, the resonant condition (50). We can replace this in equation (48) and write the approximate expression

$$I \simeq -\int \frac{\mathrm{d}k_\perp/(2\pi)^2}{(\partial u/\partial k_1)_R}\int \frac{\mathrm{d}k_1}{2\pi} g(k_1, k_\perp)\frac{\partial N_0/\partial k_1}{k_1 - k_R}. \tag{52}$$

We recognize, in the last integral, the well-known form

$$\int \frac{h(z)}{z - z_0}\,\mathrm{d}z = \mathcal{P}\int \frac{h(z)}{z - z_0}\,\mathrm{d}z + \mathrm{i}\pi h(z_0) \tag{53}$$

where the first term on the right-hand side denotes the principal part of the integral. This means that we have

$$I = -(I_1 + \mathrm{i}I_2) \tag{54}$$

with

$$I_1 = \int \frac{\mathrm{d}k_\perp/(2\pi)^2}{(\partial u/\partial k_1)_R} \mathcal{P} \int \frac{\mathrm{d}k_1}{2\pi} g(k_1, k_\perp) \frac{\partial N_0/\partial k_1}{k_1 - k_R} \tag{55}$$

and

$$I_2 = \frac{1}{8\pi^2} \int \frac{g(k_R, k_\perp)}{(\partial u/\partial k_1)_R} \left(\frac{\partial N_0}{\partial k_1}\right)_R \mathrm{d}k_\perp. \tag{56}$$

The dispersion relation becomes $\Omega^2 - q^2c^2 = a(I_1 + \mathrm{i}I_2)$. Defining $\Omega = \Omega_r + \mathrm{i}\gamma$, we obtain the modified dispersion relation that determines the real part of the frequency Ω_r, for a given q

$$\Omega_r^2 - q^2c^2 = aI_1 \tag{57}$$

and from the imaginary part, we obtain the damping rate

$$\gamma = \frac{aI_2}{2\Omega_r} \simeq \frac{aI_2}{2qc} \tag{58}$$

where we have assumed that $aI_1 \ll q^2c^2$. We conclude that the existence of a photon background introduces a damping rate on the gravitational wave that is proportional to the derivative of the photon density number, calculated at the resonant value of k_1:

$$\gamma \propto G \left(\frac{\partial N_0}{\partial k_1}\right)_R. \tag{59}$$

For a thermal population of photons this derivative is always negative, which means that we will have a wave damping induced by the resonant photon population. This is nothing but the photon Landau damping of the gravitational wave.

Note that the quantity aI_1 is extremely small and leads to a negligible deviation of the phase velocity of the gravitational wave with respect to its vacuum value c. It has been argued [5] that the quantity I_1 is always positive for a thermal distribution N_0. According to equation (57) this would mean that the phase velocity of the gravitational wave is always superluminal in a background homogeneous and stationary gas, $v_\phi = \Omega_r/q \geqslant c$. As a result, no such thing as a resonant interaction of this wave with the photon gas would be possible.

However, a gravitating homogeneous gas can never be stationary. Let us assume that apart from the photon gas, there is also in addition a small amount of dust, which is gravitational contracting. In that case, it can easily be shown that we have $v_\phi < c$ [6]. Such a correction leads to the replacement of equation (57) by

$$\Omega_r^2 - q^2c^2 = aI_1 - \tfrac{4}{3}\kappa\rho \tag{60}$$

where ρ is the density of the low-density dust gas, and where the dust pressure is assumed as negligible.

A similar result of the subluminal phase velocities could also be obtained if, instead of dust, we had considered the internal structures of the molecules of a low-density background gas [6]. This means that in a large variety of situations compatible with astrophysical models, we can assume that $v_\phi < c$, in which case the resonant interactions with the photons will lead to photon Landau damping as described above.

7. Photon beam instability

We know that the two possible polarization states for a gravitational wave propagating along x^1 are

$$\mathrm{e}^{23} = 1 \qquad \mathrm{e}^{22} = -\mathrm{e}^{33} = \frac{1}{\sqrt{2}} \tag{61}$$

Let us concentrate on the second polarization state. We have

$$\mathrm{e}^{ij*}k_ik_j = \frac{1}{2}k_2^2 \qquad \mathrm{e}^{\mu\nu}\Omega_{\mu\nu} = \frac{1}{\sqrt{2}} \qquad \Omega_{22} = \frac{k_2^2}{2k^2}. \tag{62}$$

We then get $g(k_1, k_2) = \left(k_2^2/4k^3\right)$. Let us assume an intense photon beam with finite width, as described by

$$N_0(k_1, k_2) = N_0 \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{k_2}{k_1}\right)^2\right\}. \tag{63}$$

This is similar to the distribution associated with a laser pulse near the focal region, where $\sigma = \Delta\theta$ would be the focal angle. Using $(\partial u/\partial k_1)_R = k_2^2 c/k^3$, we calculate the integral I_2, defined by equation (56). The result is

$$I_2 \simeq \frac{1}{5\pi^2}\frac{c}{k}N_0\sigma^5. \tag{64}$$

The growth rate of the kinetic beam instability induced by the photon beam is then given by

$$\gamma = \frac{a}{10\pi^2}\frac{N_0c^2}{\omega\Omega}\sigma^5. \tag{65}$$

Here, $a \propto G$ and σ are very small quantities, and N_0 and ω can be very large ones. We should also note that, in order to obtain a resonant condition $k_1 = k_R$, we must have a non-negligible dispersive contribution to the gravitational wave, in order to allow for the gravitational wave phase velocity to be less than c. This means that in order to have a clear answer on the possibility of resonance in a specific environment, which could be, for instance, a laboratory with ultra-intense beams, or a dusty plasma near an astrophysical object, a careful numerical investigation will be needed.

8. Cold photon beam

We return to the kinetic dispersion relation (41), and assume that the photon beam propagates along a well-defined direction x^1 without significant angular spreading. This means that we can write

$$N_0(k_1, k_2, k_3) = (2\pi)^2 N_0(k_1)\delta(k_2)\delta(k_3). \tag{66}$$

Integrating with respect to k_2 and k_3, we get

$$\Omega^2 - q^2c^2 = -a\int g(k_1)\frac{q_1\partial N_0/\partial k_1}{(\Omega - q_1 v(k_1))}\frac{\mathrm{d}k_1}{2\pi}. \tag{67}$$

Here, we note that the photon velocity in vacuum is constant ($v(k_1) = c$), which allows us to write, after an integration by parts,

$$\Omega^2 - q^2c^2 = \frac{aq_1}{\Omega - q_1c}\int g\prime(k_1)N(k_1)\frac{\mathrm{d}k_1}{2\pi}. \tag{68}$$

The unstable gravitational waves will propagate at an angle θ with respect to the photon beam, meaning that $q_1 = q\cos\theta$. Now we assume that the photon beam is not only unidirectional but also mono-energetic. In other words, we assume a cold beam of the form

$$N_0(k_1) = 2\pi N_0\delta(k_1 - k). \tag{69}$$

We also note that $g\prime(k) = -\sin^2\theta/k^2$. Equation (68) is then reduced to

$$1 - \frac{q^2c^2}{\Omega^2} + \frac{a_k(\theta)qc}{\Omega^2(\Omega - qc\,\cos\theta)} = 0 \tag{70}$$

with

$$a_k(\theta) = \frac{a}{ck^2}N_0\,\cos\theta\,\sin^2\theta. \tag{71}$$

In order to find the instability criteria, let us consider

$$\Omega = qc\,\cos\theta + \eta \tag{72}$$

with $\eta \ll \Omega$ and $\cos\theta$ not too small. Replacing this in equation (70) we obtain

$$\eta[2\eta qc - q^2c^2(1 - \cos^2\theta)] + a_k(\theta)qc = 0. \tag{73}$$

Assuming $\theta \ll 1$, we obtain the following solution:

$$\eta = \tfrac{1}{2}qc\theta^2 + \tfrac{1}{2}\sqrt{q^2c^2\theta^4 - 2a_k(\theta)}. \tag{74}$$

Noting that $a_k(\theta) \propto \theta^2$, we see that for small angles we can have

$$\theta^4 \leqslant \frac{2a_k(\theta)}{q^2c^2}. \tag{75}$$

This gives the instability criterion. According to equation (71), this can also be written as

$$0 < \theta \leqslant \left(\frac{2aN_0}{k^2q^2c^3}\right)^{1/2}. \tag{76}$$

The growth rate is then approximately written as

$$\eta = \mathrm{i}\sqrt{2a_k(\theta)} \propto \mathrm{i}\theta\sqrt{G}. \tag{77}$$

This result shows that for a cold photon beam, the growth rate of the gravitational waves propagating at angle θ with respect to the beam is proportional to $\sqrt{G}$. This is much larger than the kinetic growth rate associated with the inverse Landau damping, which is proportional to G. We also conclude that for a purely one-dimensional configuration where $\theta = 0$, the instability can never occur: the larger growth rates correspond to a small (but not exactly zero) angle θ. This angular constraint is due to the transverse character of gravitational wave polarization.

9. Conclusions

In this paper we have examined the possibility of the occurrence of instabilities of gravitational waves due to the presence of photon beams. This was done by using the methods of photon kinetic theory, where the electromagnetic radiation is treated as a fluid and its spacetime evolution is described by a kinetic equation. We have considered linear gravitational waves propagating in a (nearly) vacuum background and in the presence of the photon gas.

We have shown that under appropriate conditions, Landau damping of gravitational waves due to the photon gas can eventually take place, in agreement with previous study [6]. Our formulation is, however, more general in the sense that we assume a generic photon distribution and not just a thermal radiation spectrum. This leads us to formulate new questions concerning the possible kinetic instability produced by gaussian photon beams, which can be seen as an inverse Landau damping. Finally, the case of a cold and mono-energetic photon beam is also considered. It is shown that the resulting instability growth rate is much higher than that of the kinetic instability.

It should also be mentioned that such instabilities are not likely to occur in the context of laser-beam gravitational wave detectors, because the associated laser beam energy densities are quite low and the instability growth rates discussed above will always be negligible. However, with future ultra-intense laser systems working well above the Peta–Watt power level, the observation of such instabilities will eventually become possible.

In this paper, the dispersion effects of gravitational waves, in particular those leading to a subluminal phase velocity, were not discussed in a self-consistent way, because this would be too cumbersome. A detailed discussion of the dispersion effects that can occur in a plasma will be given in a future work.

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